09/16/20 Contragredient Repn.

I) G tdlc, $\mathcal{H}_G$. [Vin08]

II) Idempotented alg. [Bum, 3.4]

III) Main thm [God, 6] [Bum, 4.2] [BZ76]

- π irr. $\Leftrightarrow$ $\hat{\pi}$ irr. for GL_n. i.e, charactr by an adm. rep'n

Def'n. (π, V) be smth. repn of G_F. $\mathrm{Map}(V, \mathbb{C}) \circlearrowleft G_F$. $\langle x, f\rangle := f(x)$.

- $\hat{V} :=$ subspc of smth. fnls.
$$\{ f \in \mathrm{Map}(V, \mathbb{C}) : \exists\, \mathcal{U} \overset{c}{\subseteq}_{\text{open}} G,\ \overset{ev.}{\langle \pi(g) v, f\rangle} \underset{\langle v, f\rangle}{\overset{\shortparallel}{}}\ \forall g \in \mathcal{U}, v \in V \}$$

- $\hat{V}$ is also a smth. repn of G_F

Thm [JL, 2.18] [God, 6] let (π, V) be an inf. irr. adm. of G_F. ω its central quasi char.

$$(\hat{\pi}, \hat{V}) \simeq (\omega^{-1} \otimes \pi, V).$$

Def'n: let G be tdlc, Z center of G. π admits central quasi char if $\exists\ \chi : Z \to \mathbb{C}^\times$ s.t. $\pi(g) = \chi(g)\ \forall g \in Z$.

Ex. • irr. adm. repn. (π, V) of G_F, its center acts by scalar. Here (π, V) admits a central quasichar ω.

I. Hecke Algebra.

G tdlc., unimodular.

<u>Def'n</u>. • $\mathrm{Map}_c^\infty(G, \mathbb{C})$ ass. with alg. str.

$$f_1 * f_2 (h) := \int_G f_1(hg^{-1}) f_2(g)\, dg.$$

• dg is a fixed Haar (G unimodular)

• Denote $\mathcal{H}_G$ for this alg.

II. Idempotented alg. [Bum, 3.4]

- ring are not. necessarily unital or commutative.
- idem: $e^2 = e$.
- po on idem. $f \leq e \iff fe = ef = f$.

<u>Def'n</u>. An idem. alg. over a field k, consists of

- (H, E) H is alg. over k, E is coll'n idem in H.
- E to be cofiltered. i.e. $e_1, e_2 \in E$, $\exists f \in E$ $e_1, e_2 \leq f$.
- $\forall x \in H$, $\exists e \in E$, $ex = xe = x$.

<u>Prop</u>: $\mathcal{H}_G$ is an idemp. alg.

<u>Pf</u>: • E consists of elements of fun $\epsilon_U := \frac{1}{\mathrm{vol}(U)} 1_U$. U cpt open G.

•

$$\epsilon_U * \epsilon_U = \epsilon_U$$

- $V \subseteq U$.

$$e_V * e_U\ (h) = \int_G 1_V(hg^{-1})\, 1_U(g)\, dg$$
$$= \int_U 1_V(hg^{-1})\, dg$$
$$= e_V(h) \qquad (*)$$

- with $(*)$ can check E is cofiltered.

- [std. chk]

IIa Modules of idempotented alg. M is a H-module.

Not'n: • $H[e] := eHe$., $e \in H[e]$ is a unit., $e \in$ Idem of H.

• $M[e]$ denote the $H[e]$ module eM.

Def'n: 1) M is smooth iff

$$\varinjlim_{e \in E} M[e] \xrightarrow{\sim} M \qquad \text{(this is just the union)}$$

equivalently: $\forall x \in M$, $\exists e \in E$ s.t. $ex = x$

2) M is adm. if it is smooth.
& $M[e]$ is f.d. $\forall e \in E$

$G_F := GL_2(F)$

IIb. $\mathcal{H}_{G_F}$ modules $\longleftrightarrow$ repns of G_F

Thm: [JL, p25] (G_F, but works for td.l.c. G)

- There is a bijn.

$$\begin{tikzcd}
\{\text{smooth repns of } G_F\} \arrow[r] & \{\text{smth modules of } \mathcal{H}_{G_F}\} \\
\{\text{adm. repns of } G_F\} \arrow[r] \arrow[u, hook] & \{\text{adm. modls of } \mathcal{H}_{G_F}\} \arrow[u, hook]
\end{tikzcd}$$

- (π, V) smooth repn of G_F, $U \subseteq_{\mathbb{C}} V$
Then U is stable under G_F $\Leftrightarrow$ it stable under $\mathcal{H}_{G_F}$.

Pf.

Step 1: lhs to rhs. Define a $\mathcal{H}_G$ action on V. given (π, V) $f \in \mathcal{H}_G$.

$$\pi f \, v := \int_G f(g)\pi(g)\, v \, dg.$$

Step 1a: this action is well defined: Integral is a finite sum.

$g \to f(g)\pi(g)v$ is loc. constant.

- $g \mapsto \pi(g)v$. is loc. cst. $G \to V$ [smooth]
- $f \in \mathrm{Map}_c^\infty(G, \mathbb{C})$, then f'n is $\mathrm{Map}_c^\infty(G, V)$

$\therefore$ Integral is a finite sum.

Step 1b. $\pi(f_1 * f_2) = \pi f_1 \circ \pi f_2$

$\therefore$ $\pi : \mathcal{H}_G \to \mathrm{End}_{\mathbb{C}} V$, is $\mathbb{C}$-alg. homo. i.e. $\mathcal{H}_G$-module.

Step 1c. This smth module. let $v \in V$

- $\exists \xi \in \mathcal{H}_G$ s.t. $\pi(\xi) v = v$.

- By smthness, $\exists \mathcal{U}, \subseteq_{\text{open}} G$ stabilises v.
- $\exists \mathcal{U}' \subseteq_{\text{cpt open}} \mathcal{U}$, $\pi(e_{\mathcal{U}'}) \cdot v = v$.

Step 2. define the map. $\mathcal{H}_G$-module V. let $g \in G$. let $v \in V$.

- write $v = \sum_{i=1}^{n} \pi f_i \cdot v_i$ $\quad f_i \in$ idem, $v_i \in V$. $\pi : \mathcal{H}_G \to \mathrm{End}_{\mathbb{C}} V$. ,

(can choose $i=1$, by defn of smooth)

- $\pi(g) \cdot v := \sum_{i=1}^{n} \pi(\lambda(g) f_i) v_i$ $\quad$ λ is left reg action G on $\mathcal{H}_G$ $\left(\begin{array}{c} \lambda(g) f(h) = f(g^{-1}h) \\ \forall f \in \mathcal{H}_G \end{array} \right)$.

Step 3. standard checks $\square$.

II. Contragred. mod. , H an idem. alg. over k

Ctx: • let ι be an anti involution on H

Def'n. let M be a H-module.

- $M^* := \mathrm{Map}(M, k) \curvearrowleft H$ $\quad \langle x, rf \rangle := \langle r^{\iota} x, f \rangle$
- $\tilde{M} := \{ f \in \mathrm{Map}(M, k) : f$ has a stabilizer in $E \}$.

$\tilde{M}$ is the contragr. module of M.

- $\sim$ also resp. adm. mod.

Prop: $\sim$: adm. H-mod $\longrightarrow$ Adm H-mod^{op}

$M \longmapsto \tilde{M}$

1) Functor is exact.

2) $\sim^2$ is nat. iso id. i.e. $\tilde{\tilde{M}} \cong M$. as H-module.

Prop: M, N are two H-modules. β is a non zero bil. on $M \times N$.

1. $M \longrightarrow N^*$ $\quad m \mapsto \beta(m, -)$ factors $\tilde{N}$.

$$\begin{tikzcd} M \arrow[r] \arrow[dr] & N^* \\ & \tilde{N} \arrow[u] \end{tikzcd}$$

2. If M, N are simple H modules. β is non deg.

3. If β is non deg. then $M \simeq \tilde{N}$.

Pf. 1+2 easy checks. 3 uses 2. and exactness of $\sim$.

Thm [JL, 2.18] [Bod, 6] let (π, V) be an inf. irr. adm. of G_F.
ω its central quasi-character.

$$(\hat{\pi}, \hat{V}) \simeq (\omega^{-1} \otimes \pi, V).$$

Pf: strategy define a pairing that is G_F equivariant. $\qquad S(F^\times) \quad w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$V \simeq V_0 + w V_0$.

$$\beta : V \times V \longrightarrow \mathbb{C}.$$

via π. $\qquad$ action is via $\omega^{-1} \otimes \pi$.

• [Bum, 3.4] $\Rightarrow (\hat{\pi}, \hat{V}) \simeq (\omega^{-1} \otimes \pi, V)$. $\qquad$ $\square$.